Math 201 — Fall 2008–09
Calculus and Analytic Geometry III, sections 21–23
Quiz 2, December 2 — Duration: 70 minutes

GRADES (each problem is worth 12 points):

1	2	3	4	5	6	TOTAL/72

YOUR NAME:

YOUR AUB ID#:

PLEASE CIRCLE YOUR SECTION:

Section 21	Section 22	Section 23
Recitation M 11	Recitation M 9	Recitation M 1
Ms. Itani	Professor Makdisi	Ms. Itani

INSTRUCTIONS:
1. Write your NAME and AUB ID number, and circle your SECTION above.
2. Solve the problems inside the booklet. Explain your steps precisely and clearly to ensure full credit. Partial solutions will receive partial credit.
3. You may use the back of each page for scratchwork OR for solutions. There are three extra blank sheets at the end, for extra scratchwork or solutions. If you need to continue a solution on another page, INDICATE CLEARLY WHERE THE GRADER SHOULD CONTINUE READING.
4. Open book and notes. NO CALCULATORS ALLOWED. Turn OFF and put away any cell phones.

GOOD LUCK!

An overview of the exam problems. Each problem is worth 12 points.
Take a minute to look at all the questions, THEN
solve each problem on its corresponding page INSIDE the booklet.

1. Let $f(x,y,z) = e^{xy} - xz$.

(2 pts) a) Find the gradient $\vec{\nabla} f$.

(5 pts) b) Let $P(t)$ be a moving point with position $\vec{r}(t)$ and velocity $\vec{v}(t)$. Assume we know that at time $t=0$, we have $\vec{r}(0) = (1,1,2)$ and $\vec{v}(0) = (-1,3,1)$. Find $\left.\frac{d}{dt} f(\vec{r}(t))\right|_{t=0}$.

(5 pts) c) Let $S = \{(x,y,z) \in \mathbf{R}^3 \mid f(x,y,z) = -1\}$ be the level set of f passing through the point $P_0(2,0,1)$. Find the equation of the tangent plane to S at P_0.

2. (5 pts) a) Sketch the curve $r = \sin^2\theta$ in polar coordinates.

(2 pts) b) Write a parametrization of your curve in the form $\vec{r}(\theta) = (x(\theta), y(\theta))$, for $0 \le \theta \le 2\pi$. (Yes, I want you to use θ instead of t as the time variable.)

(5 pts) c) When $\theta = \pi/4$, find the position vector $\left.\vec{r}\right|_{\theta=\pi/4}$ and the velocity vector $\left.\vec{v}\right|_{\theta=\pi/4}$. Use $\vec{v}$ to deduce the speed of the moving point at $\theta = \pi/4$.

3. Let $f(x,y) = \ln(x^2+y)$.

(4 pts) a) Sketch the domain of f and **briefly** indicate why the domain is **not** closed.

(4 pts) b) Find the directional derivative of f at the point $P_0(3,1)$ in the direction of the vector $\vec{A} = (3,4)$. (Suggestion: do not simplify fractions; for example, keep $40/70$ the way it is, and do not reduce it to $4/7$.)

(4 pts) c) Approximately how much is $f(3.03, 1.04)$? This can be done either using (b) or by a direct calculation. You may choose whichever method you prefer.

4. (3 pts) a) Use integration by parts twice to show that if n is a constant, then

$$\int x^2 \sin nx\,dx = \frac{-x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3} + C.$$

(7 pts) b) Let $f(x)$ be given by $f(x) = \begin{cases} 0, & \text{if } -\pi < x < 0 \\ x^2, & \text{if } 0 \le x \le \pi. \end{cases}$ Extend f periodically to have period 2π. Your job is to **sketch the graph***of f and to **compute ONLY the coefficients** b_n in the Fourier series $f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$. $\boxed{\text{* for } x \in [-3\pi, 3\pi]}$

(2 pts) c) At what points x does the Fourier series of f **not** converge to $f(x)$? What is the value of the Fourier series at those points?

5. We wish to find the maximum and minimum of $f(x,y) = x(y^2-1)$ in the half disk D defined by $D = \{(x,y) \in \mathbf{R}^2 \mid x^2+y^2 \le 10, y \ge 0\}$. (Note that D is closed and bounded, so f attains a maximum and minimum on D.)

(3 pts) a) Show that f has exactly two critical points in the plane $\mathbf{R}^2$, but that only **one** of these points belongs to D.

(3 pts) b) Test the top part of the boundary of D (this means the semicircle $x^2+y^2 = 10$ where $y \ge 0$) for possible maxima and minima of f. Parametrize the boundary in the following way: $P(t) = (t, \sqrt{10-t^2})$. Remember to specify the range of t for your parametrization.

(3 pts) c) Test the bottom part of the boundary (this means the line segment) for possible maxima and minima.

(3 pts) d) Make a table of values that includes the points you found above, as well as the corners, and deduce the maximum and minimum values of f, as well as the points where they are attained.

6. (3 pts) a) State the first three terms in the Maclaurin series of $(1+u)^{1/3}$. Express your answer in the form $(1+u)^{1/3} = a + bu + cu^2 + O(u^3)$. (In other words, find specific numbers a, b, and c.)

(4 pts) b) Show that $x + y = O(\Delta s)$, where $\Delta s = \sqrt{x^2+y^2}$. Use this to show that

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^3}{x^2+y^2} = 0.$$

(5 pts) c) Use the above results to find the limit

$$\lim_{(x,y)\to(0,0)} \frac{(1+x+y)^{1/3} - 1 - x/3 - y/3 + 2xy/9}{x^2+y^2}.$$

1. Let $f(x,y,z) = e^{xy} - xz$.

(2 pts) a) Find the gradient $\vec{\nabla} f$.

(5 pts) b) Let $P(t)$ be a moving point with position $\vec{r}(t)$ and velocity $\vec{v}(t)$. Assume we know that at time $t=0$, we have $\vec{r}(0) = (1,1,2)$ and $\vec{v}(0) = (-1,3,1)$. Find $\frac{d}{dt} f(\vec{r}(t))\Big|_{t=0}$.

(5 pts) c) Let $S = \{(x,y,z) \in \mathbf{R}^3 \mid f(x,y,z) = -1\}$ be the level set of f passing through the point $P_0(2,0,1)$. Find the equation of the tangent plane to S at P_0.

a) $\vec{\nabla} f = (f_x, f_y, f_z) = \boxed{\left(ye^{xy} - z,\ xe^{xy},\ -x\right)}$

b) $\frac{d}{dt} f(\vec{r}(t))\Big|_{t=0} = \vec{\nabla} f\Big|_{\vec{r}(0)} \cdot \frac{d\vec{r}}{dt}\Big|_{t=0}$ by the chain rule

$= \vec{\nabla} f\Big|_{(x,y,z)=(1,1,2)} \cdot (-1,3,1)$

$= (e-2, e, -1) \cdot (-1,3,1)$

$= -e + 2 + 3e - 1 = \boxed{2e+1}$

c) $\vec{\nabla} f\Big|_{P_0} = \vec{\nabla} f\Big|_{(2,0,1)} = (-1, 2e^0, -2) = (-1,2,-2)$

the tangent plane is $\perp$ $\vec{\nabla} f\big|_{P_0}$.

we have: $T(x_T, y_T, z_T) \in$ tangent plane

$\Leftrightarrow \overrightarrow{P_0T} \cdot \vec{\nabla} f\big|_{P_0} = 0$

$\Leftrightarrow (x_T - 2, y_T - 0, z_T - 1) \cdot (-1,2,-2) = 0$

$\Leftrightarrow -x_T + 2 + 2y_T - 2z_T + 2 = 0$

$\Leftrightarrow \boxed{-x_T + 2y_T - 2z_T = -4}$

2. (5 pts) a) Sketch the curve $r = \sin^2\theta$ in polar coordinates.

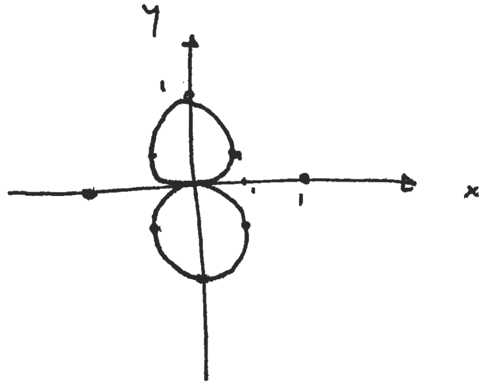
(2 pts) b) Write a parametrization of your curve in the form $\vec{r}(\theta) = (x(\theta), y(\theta))$, for $0 \le \theta \le 2\pi$. (Yes, I want you to use θ instead of t as the time variable.)

(5 pts) c) When $\theta = \pi/4$, find the position vector $\vec{r}\,\big|_{\theta=\pi/4}$ and the velocity vector $\vec{v}\,\big|_{\theta=\pi/4}$. Use $\vec{v}$ to deduce the speed of the moving point at $\theta = \pi/4$.

a)

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$\sin^2\theta$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0

$\left(\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \text{ so } \sin^2\frac{\pi}{4} = \frac{1}{2}\right)$



(note tangency between the curve and the x-axis)

b) $\vec{r}(\theta) = (x(\theta), y(\theta)) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$

so $\boxed{\vec{r}(\theta) = (\sin^2\theta\cos\theta, \sin^3\theta)}$

c) $\vec{v} = \frac{d\vec{r}}{d\theta} = (2\sin\theta\cos\theta \cdot \cos\theta - \sin^2\theta \cdot \sin\theta, 3\sin^2\theta\cos\theta)$

$= (2\sin\theta\cos^2\theta - \sin^3\theta, 3\sin^2\theta\cos\theta)$

at $\theta = \frac{\pi}{4}$, $\sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$, so

$\vec{r}\left(\frac{\pi}{4}\right) = \left(\frac{1}{2}\cdot\frac{1}{\sqrt{2}}, \left(\frac{1}{\sqrt{2}}\right)^3\right) = \boxed{\left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)}$ position at $\theta = \pi/4$

$\vec{v}\left(\frac{\pi}{4}\right) = \left(2\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{2} - \frac{1}{2\sqrt{2}}, 3\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{2}}\right) = \boxed{\left(\frac{1}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}\right)}$ velocity " "

$\left|\vec{v}\left(\frac{\pi}{4}\right)\right| = \sqrt{\left(\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{3}{2\sqrt{2}}\right)^2} = \sqrt{\frac{1}{8} + \frac{9}{8}} = \sqrt{\frac{10}{8}} = \sqrt{\frac{5}{4}} = \boxed{\frac{\sqrt{5}}{2}}$ speed at $\theta = \frac{\pi}{4}$

3. Let $f(x, y) = \ln(x^2 + y)$.

(4 pts) a) Sketch the domain of f and **briefly** indicate why the domain is **not** closed.

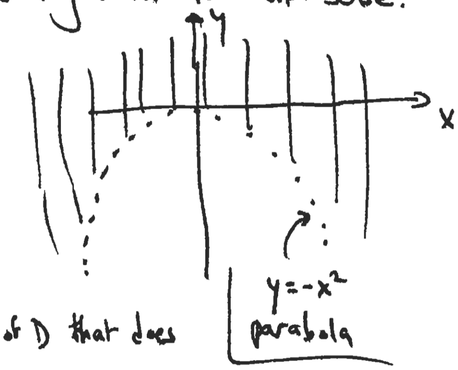
(4 pts) b) Find the directional derivative of f at the point $P_0(3, 1)$ in the direction of the vector $\vec{\mathbf{A}} = (3, 4)$. (Suggestion: do not simplify fractions; for example, keep 40/70 the way it is, and do not reduce it to 4/7.)

(4 pts) c) Approximately how much is $f(3.03, 1.04)$? This can be done either using (b) or by a direct calculation. You may choose whichever method you prefer.

a) Domain of f: requires $x^2+y > 0$ for the logarithm to make sense.

so $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y>0\}$

$= \{(x,y) \in \mathbb{R}^2 \mid y > -x^2\}$



D is shaded in the figure.

It is not closed because $(0,0)$ (AMONG OTHERS) is a boundary pt of D that does NOT belong to D.

b) $\vec{\nabla} f = \left(\frac{2x}{x^2+y}, \frac{1}{x^2+y}\right)$ so $\vec{\nabla} f\Big|_{P_0} = \left(\frac{6}{9+1}, \frac{1}{9+1}\right) = \left(\frac{6}{10}, \frac{1}{10}\right)$

$\vec{A}$ is not a unit vector, but $\vec{u} = \frac{\vec{A}}{|\vec{A}|}$ is a unit vector in the same direction.

thus $\vec{u} = \frac{(3,4)}{\sqrt{3^2+4^2}} = \frac{(3,4)}{5} = \left(\frac{3}{5}, \frac{4}{5}\right)$.

Finally, $D_{\vec{u}} f\Big|_{P_0} = \vec{\nabla} f\Big|_{P_0} \cdot \vec{u} = \left(\frac{6}{10}, \frac{1}{10}\right) \cdot \left(\frac{3}{5}, \frac{4}{5}\right)$

$= \frac{18}{50} + \frac{4}{50} = \boxed{\frac{22}{50}}$.

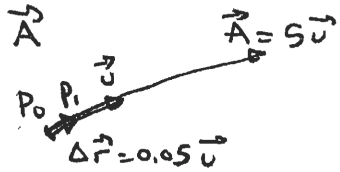
c) 1st way $P_0(3,1)$ $P_1(3.03, 1.04)$, displacement $\Delta\vec{r} = \vec{P_0P_1} = (0.03, 0.04)$

$= 0.01\vec{A}$

so $\Delta\vec{r} \parallel \vec{A}$ (so $\Delta\vec{r} = 0.05\vec{u}$)

and $\Delta s = |\Delta\vec{r}| = 0.05$ (easy)

we have moved 0.05 units in the direction of $\vec{u}$.



Thus $\Delta f \approx D_{\vec{u}} f\Big|_{P_0} \Delta s = \frac{22}{50} \cdot 0.05 = 22 \times 10^{-3} = 0.022$

$f(P_1) = f(P_0) + \Delta f = \ln(3^2+1) + \Delta f \approx \boxed{\ln 10 + 0.022}$.

2nd way use $\Delta\vec{r} = (0.03, 0.04)$ so $\Delta f \approx \vec{\nabla} f\Big|_{P_0} \cdot \Delta\vec{r}$

$= \left(\frac{6}{10}, \frac{1}{10}\right) \cdot (0.03, 0.04) = 0.018 + 0.004$

$= 0.022$

then you get the same answer $f(P_1) = \ln 10 + \Delta f \approx \boxed{\ln 10 + 0.022}$

4. (3 pts) a) Use integration by parts twice to show that if n is a constant, then

$$\int x^2 \sin nx\, dx = \frac{-x^2 \cos nx}{n} + \frac{2x \sin nx}{n^2} + \frac{2 \cos nx}{n^3} + C.$$

(7 pts) b) Let $f(x)$ be given by $f(x) = \begin{cases} 0, & \text{if } -\pi < x < 0 \\ x^2, & \text{if } 0 \le x \le \pi. \end{cases}$ Extend f periodically to have period 2π. Your job is to **sketch the graph*** of f and to **compute ONLY the coefficients** b_n in the Fourier series $f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$.

(2 pts) c) At what points x does the Fourier series of f **not** converge to $f(x)$? What is the value of the Fourier series at those points?

$\boxed{\text{* for } x \in [-3\pi, 3\pi]}$

$\boxed{\text{this is } d(x^2)}$

a) $$\int x^2 \sin nx\,dx = \int x^2\, d\left(\frac{-\cos nx}{n}\right) \overset{\boxed{\text{int. by parts}}}{=} -x^2 \frac{\cos nx}{n} + \int \frac{\cos nx}{n} \cdot 2x\,dx$$

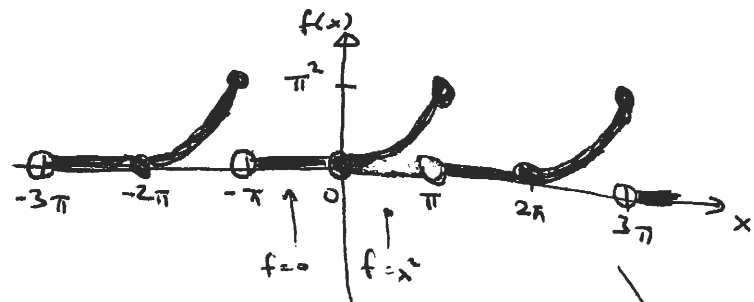
$$= -\frac{x^2 \cos nx}{n} + \frac{2}{n}\int x\cos nx\,dx = -\frac{x^2\cos nx}{n} + \frac{2}{n^2}\int x\, d(\sin nx)$$

$$\overset{\boxed{\text{int. by parts}}}{=} -\frac{x^2 \cos nx}{n} + \frac{2}{n^2}\left[x\sin nx - \int \sin nx\,dx\right] = -\frac{x^2\cos nx}{n} + \frac{2}{n^2}x\sin nx + \frac{2}{n^2}\frac{\cos nx}{n}$$

$$= \frac{-x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3} \quad (+C)$$

indefinite integral!

b)



$$b_n = \frac{1}{\pi}\int_{x=-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\cancel{\int_{x=-\pi}^{0} 0 \sin nx\,dx} + \int_{x=0}^{\pi} x^2 \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_{x=0}^{\pi} = \frac{1}{\pi}\left(\frac{-\pi^2\cos n\pi}{n} + 0 + \frac{2\cos n\pi}{n^3} + 0 - 0 - \frac{2}{n^3}\right)$$

so

$$\boxed{b_n = \frac{1}{\pi}\left(\frac{-\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right)}$$

(using $\cos n\pi = (-1)^n$)

c) the Fourier series converges to $f(x)$ wherever $\begin{cases} f \text{ is continuous \&} \\ f \text{ has a left \& right derivative.} \end{cases}$

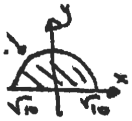
This is at all points <u>EXCEPT</u> at $x = \pi,\ x = 3\pi,\ x = 5\pi, \ldots$

$x = -\pi,\ x = -3\pi,\ x = -5\pi$, etc..

at such points the Fourier series converges to $\dfrac{f(x^+) + f(x^-)}{2} = \dfrac{0 + \pi^2}{2} = \boxed{\dfrac{\pi^2}{2}}$

$x = \pi + 2k\pi$

$k \in \mathbb{Z}$



5. We wish to find the maximum and minimum of $f(x,y) = x(y^2-1)$ in the half disk D defined by $D = \{(x,y) \in \mathbf{R}^2 \mid x^2+y^2 \le 10, y \ge 0\}$. (Note that D is closed and bounded, so f attains a maximum and minimum on D.)

(3 pts) a) Show that f has exactly two critical points in the plane $\mathbf{R}^2$, but that only **one** of these points belongs to D.

(3 pts) b) Test the top part of the boundary of D (this means the semicircle $x^2+y^2=10$ where $y \ge 0$) for possible maxima and minima of f. Parametrize the boundary in the following way: $P(t) = (t, \sqrt{10-t^2})$. Remember to specify the range of t for your parametrization.

(3 pts) c) Test the bottom part of the boundary (this means the line segment) for possible maxima and minima.

(3 pts) d) Make a table of values that includes the points you found above, as well as the corners, and deduce the maximum and minimum values of f, as well as the points where they are attained.

a) $\vec{\nabla} f = (y^2-1, 2xy)$. Critical points occur when $y^2-1=0$ AND $2xy=0$ so $y = \pm 1$, and in particular $y \neq 0$. Thus $2xy=0 \Leftrightarrow x=0$ since $y \neq 0$.
& we have $x=0$, $y=\pm 1$ meaning $\boxed{P_0\ (0,1) \in D}$

$Q_0\ (0,-1) \notin D$ · Q_0 ignored

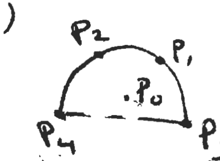
b) $P(t) = (t, \sqrt{10-t^2})$ $t \in [-\sqrt{10}, +\sqrt{10}]$
so $f(P(t)) = t(10-t^2-1) = t(9-t^2) = 9t - t^3$. Study its variation on $[-\sqrt{10}, +\sqrt{10}]$. First, the derivative w.r.t. t is $9-3t^2$, which vanishes at $\left\{ t = \pm\sqrt{3} \right\}$, corresponding to the points $\boxed{P_1\ (\sqrt{3}, \sqrt{7}),\ P_2\ (-\sqrt{3}, \sqrt{7})}$
$\left(\Rightarrow \sqrt{10-t^2} = \sqrt{7} \right)$

Second, there are the endpoints $t = \pm\sqrt{10}$ (corresponding to the corners) $\boxed{P_3\ (\sqrt{10}, 0),\ P_4\ (-\sqrt{10}, 0)}$.

c) here, parametrize by $P(t) = (t, 0)$ & $f(P(t)) = t(0-1) = -t$
derivative w.r.t. t is -1, so there are NO candidates for maxima or minima on the segment, except possibly at the endpoints (this gives us the corners, P_3 & P_4, again).

d) table of values at all the candidate points

P	$P_0(0,1)$	$P_1(\sqrt{3},\sqrt{7})$	$P_2(-\sqrt{3},\sqrt{7})$	$P_3(\sqrt{10},0)$	$P_4(-\sqrt{10},0)$
$f(P)$	0	$\sqrt{3}\cdot 6 = 6\sqrt{3}$	$-\sqrt{3}\cdot 6 = -6\sqrt{3}$	$\sqrt{10}\cdot(-1) = -\sqrt{10}$	$-\sqrt{10}\cdot(-1) = +\sqrt{10}$



also note that $6\sqrt{3} > 6 > 4 > \sqrt{10}$
(or $(6\sqrt{3})^2 = 36 \cdot 3 = 108 > 10 = (\sqrt{10})^2$)

So the maximum value of f on D is $f(P_1) = 6\sqrt{3}$, attained at $P_1(\sqrt{3}, \sqrt{7})$
while the minimum value of f on D is $f(P_2) = -6\sqrt{3}$, attained at $P_2(-\sqrt{3}, \sqrt{7})$.

6. (3 pts) a) State the first three terms in the Maclaurin series of $(1+u)^{1/3}$. Express your answer in the form $(1+u)^{1/3} = a + bu + cu^2 + O(u^3)$. (In other words, find specific numbers a, b, and c.)

(4 pts) b) Show that $x + y = O(\Delta s)$, where $\Delta s = \sqrt{x^2+y^2}$. Use this to show that

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^3}{x^2+y^2} = 0.$$

(5 pts) c) Use the above results to find the limit

$$\lim_{(x,y)\to(0,0)} \frac{(1+x+y)^{1/3} - 1 - x/3 - y/3 + 2xy/9}{x^2+y^2}.$$

a) by the Binomial series expansion (or by computing the Taylor series),

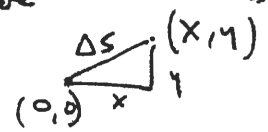
we have $(1+u)^{1/3} = 1 + \left(\frac{1}{3}\right)u + \left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\frac{u^2}{2!} + \left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\frac{u^3}{3!} + \cdots$

$$= \boxed{1 + \frac{1}{3}u - \frac{1}{9}u^2 + O(u^3)}$$

[scratch: $\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\frac{1}{2!} = \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\cdot\frac{1}{2} = -\frac{1}{9}$]

(Here $\Delta x = x$ & $\Delta y = y$)

b)



we have $(\Delta s)^2 = x^2 + y^2 \geq x^2$, so $\Delta s \geq |x|$

similarly $\Delta s \geq |y|$.

Thus $|x+y| \leq |x| + |y| \leq \Delta s + \Delta s = 2\Delta s =$ (constant) Δs

triangle inequality

This shows that $x+y$ is $O(\Delta s)$.

Hence $(x+y)^3$ is $O((\Delta s)^3)$ and $\frac{(x+y)^3}{x^2+y^2} = \frac{(x+y)^3}{(\Delta s)^2}$ is $O\frac{((\Delta s)^3)}{(\Delta s)^2}$; now cancel $(\Delta s)^2$

$\therefore$ $\boxed{\frac{(x+y)^3}{x^2+y^2} \text{ is } O(\Delta s)}$ and this shows that as $(x,y)\to(0,0)$, we have $\Delta s \to 0$, hence $O(\Delta s) \to 0$ hence $\frac{(x+y)^3}{x^2+y^2} \to 0$.

c) put $u = x+y$ in part (a) & obtain

$$(1+x+y)^{1/3} - 1 - \frac{x}{3} - \frac{y}{3} + \frac{2xy}{9} = 1 + \frac{x+y}{3} - \frac{x^2+2xy+y^2}{9} + O((x+y)^3) - 1 - \frac{x}{3} - \frac{y}{3} + \frac{2xy}{9}$$

$$= -\frac{x^2+y^2}{9} + O((x+y)^3)$$ [the other terms cancel].

Thus $$\frac{(1+x+y)^{1/3} - 1 - \frac{x}{3} - \frac{y}{3} + \frac{2xy}{9}}{x^2+y^2} = -\frac{1}{9} + O\left(\frac{(x+y)^3}{x^2+y^2}\right)$$

but this last term satisfies $O\left(\frac{(x+y)^3}{x^2+y^2}\right) \to 0$ as $(x,y)\to(0,0)$ by part (b)

$\therefore$ our desired limit is $\lim\left(-\frac{1}{9} + O\left(\frac{(x+y)^3}{x^2+y^2}\right)\right) = -\frac{1}{9} + 0 = \boxed{-\frac{1}{9}}$